

Three-dimensional interior solutions from Jordan-Brans-Dicke Theory of Gravity.

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Abstract

We study spherically symmetric solutions to the Jordan-Brans-Dicke field equations under the assumption that the space-time may possess an arbitrary number of spatial dimensions. Assuming a perfect fluid with the equation of state $p = \varepsilon\rho$, we show that there are static interior nontrivial solutions in three dimensional Jordan-Brans-Dicke gravity theory.

1 Introduction

The most studied and hence the best known alternative of classical Einstein's gravity is the scalar-tensor Jordan-Brans-Dicke (JBD) theory.[1][2].The essential feature of JBD theory is the presence of a massless scalar field to describe gravitation together with the metric. Scalar-tensor theories contain arbitrary functions of the scalar field that determine the scalar potential as a dynamical variable, the analog of gravitational permittivity is allowed to vary with space and time which defined using Newton's gravitational constant as $G = 1/\phi$.

The investigate of a JBD field equations in dimensions lower than 3+1 is interesting because it may allow to study phenomena characteristic of gravity, which have 3+1 dimensional analogues in a simplified context [3]. In 2+1 dimensions, the Riemann- Christoffel tensor is uniquely determined by the Ricci tensor, which vanishes outside the sources. Hence, spacetime of Einstein General Relativity is flat in regions devoid of matter, and test particles do not feel any gravitational field. However, in 2+1 dimensions JBD

field equations reproduces Newtonian gravity when the low energy regimen is consistently analysed [4].

In general we expect that the task of finding solutions of field equations in 3+1 dimensions to be much more involved than in 2+1. Therefore it will be useful to find a exact interior solutions to the JBD equations from 2+1 dimensions. In this paper we will show that for a static perfect fluid with the proper energy density proportional to the proper pressure this can be done.

2 Scalar-tensor theories in D-dimensions

Scalar-tensor theories are described by the following action in the Jordan frame in D -dimensional space-time is:

$$S = \frac{1}{16\pi} \int d^D x \sqrt{-g} (\phi R - \omega(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \lambda(\phi)) + S_m. \quad (1)$$

Here, R is the Ricci scalar curvature with respect to the space-time metric $g_{\mu\nu}$. We use units in wich gravitational constant $\mathcal{G}=1$ and speed of light $c=1$. The dynamics of the scalar field ϕ depends on the functions $\omega(\phi)$ and $\lambda(\phi)$. It should be mentioned that the different choices of such function give different scalar tensor theories. We restrict our discussion to the JBD theory which characterized by the functions $\lambda(\phi) = 0$ and $\omega(\phi) = \omega/\phi$, where ω is a constant.

The action principle with suitable boundary conditions gives rise to the D-dimensional field equations:

$$G_{\mu\nu} = -\frac{8\pi}{\phi} T_{\mu\nu} - \frac{\omega}{\phi^2} (\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\rho} \phi^{,\rho}) - \frac{1}{\phi} (\phi_{,\mu;\nu} - g_{\mu\nu} \phi^{;\rho}_{;\rho}), \quad (2)$$

$$\phi^{;\rho}_{;\rho} = \frac{k}{2(\omega + 1)} T, \quad (3)$$

where semi-colon denotes the covariant derivative with respect to the metric $g_{\mu\nu}$ and $T_{\mu\nu}$ is the usual energy momentum tensor which obeys the conservation equation $T_{\mu\nu;\rho} g^{\nu\rho} = 0$, and k is a function of ω and dimension D :

$$k = \frac{1 + \omega}{(D - 1) + \omega (D - 2)}. \quad (4)$$

The progress in the understanding of scalar-tensor theories of gravity is closely connected with finding and investigation of exact solutions. We assume that space-time is static and the configuration is spherically symmetric. In this case further simplification is possible; the space-time metric can be put in the curvature coordinates as:

$$ds^2 = e^{2\beta} dt^2 - e^{2\alpha} dr^2 - r^2 d\Omega_{(D-2)}^2, \quad (5)$$

where $d\Omega_{(D-2)}^2$ is the line element on a unit D-2 sphere:

$$d\Omega_{(D-2)}^2 = \left[d\theta_{(0)}^2 + \sum_{n=1}^{D-3} d\theta_{(n)}^2 \left(\prod_{m=1}^n \sin^2 \theta_{(m-1)} \right) \right]. \quad (6)$$

In the case of JBD theory it allow us to find exact solutions for the field produced by a static and isotropic source in regions devoid of sources in $D = 3$. In a suitable reference system, the metric can be written in the standard form:

$$ds^2 = e^{2\beta} dt^2 - e^{2\alpha} dr^2 - r^2 d\varphi^2, \quad (7)$$

where α and β are functions of r alone. In vacuum ($T_{\mu\nu} = 0$) Eqs. (2), (3) have the solution $g_{\mu\nu} = \eta_{\mu\nu}$ and $\phi = 1$, where $\eta_{\mu\nu}$ is the Minkowski metric tensor. On the other hand in vacuum Eqs. (2), (3) yield

$$\alpha = \alpha_0 + C_\alpha \ln \left(\frac{r}{r_0} \right), \quad (8)$$

$$\beta = \beta_0 + C_\beta \ln \left(\frac{r}{r_0} \right), \quad (9)$$

$$\phi = \phi_0 \left(\frac{r}{r_0} \right)^{C_\phi}. \quad (10)$$

Hawking [5] has pointed out that the stationary space containing a black hole is a solution of the JBD field equations if and only if it is a solution of the Einstein field equations. Thus, stationary black hole solutions in the JBD theory are the same as stationary black hole solutions in the Einstein theory. However, JBD theory can be thought of as a minimal extension of general relativity designed to properly accommodate both Mach's principle [6] and Dirac's large number hypothesis [6]. The point is that the field equations admit curved vacuum solutions and also solutions that are asymptotically flat, at the same time such solutions as vacuum, a single matter particle etc. are anti-machian and, hence, has no meaning in the absence of matter.

3 Interior solutions.

Finding exact solutions of scalar-tensor theories equations in the presence of a matter is a difficult task due to their complexity in the general case. Bruckman and Kazes [7] derive the relation between the scalar field ϕ and g_{00} :

$$\phi = e^{k\beta}, \quad (11)$$

where k arbitrary constant. They use static spacetime and the energy-momentum tensor is that of a perfect fluid with equation of state $p = \varepsilon\rho$. Using this assumption an exact solution of the field equation in 3+1 dimensions is found corresponding to a density distribution that is infinite at the origin [7]. Moreover using the relation (11) we have shown that the existence of a solution of Eqs. (2), (3) in 3+1 dimensions with equation of state $\rho = 0$ and $p \neq 0$ [8]. However, this standard tenet about the relation between ϕ and g_{00} can be false, this relation in general case is more complexity [8].

The equations (2), (3) are particularly simple with the choice space in 2+1 dimensions. In this case we get the field equations of JBD theory produced by a static and isotropic source in regions devoid of sources in $D = 3$. The above metric (7) yields the following field equations for (2), (3):

$$-\frac{\phi'}{r} + \alpha'\phi' - \beta'\phi' - \phi'' = \frac{4\pi e^{2\alpha}(2p - \rho)}{2 + \omega}, \quad (12)$$

$$\frac{\alpha'}{r} + \alpha' \beta' - \beta'^2 + \frac{\alpha' \phi'}{\phi} - \frac{\omega \phi'^2}{\phi^2} - \beta'' - \frac{\phi''}{\phi} = \frac{8\pi e^{2\alpha} (\omega p - (1 + \omega) \rho)}{(2 + \omega) \phi}, \quad (13)$$

$$r \left(\alpha' - \beta' - \frac{\phi'}{\phi} \right) = \frac{8\pi e^{2\alpha} r^2 (\omega p - (1 + \omega) \rho)}{(2 + \omega) \phi}, \quad (14)$$

$$\frac{\beta'}{r} - \alpha' \beta' + \beta'^2 + \frac{\beta' \phi'}{\phi} + \beta'' = -\frac{8\pi e^{2\alpha} (2(1 + \omega) p + \rho)}{(2 + \omega) \phi}, \quad (15)$$

where now the primes stand for derivation respect to r . Let us start with the problem of finding out the space-time and the scalar field generated by a static configuration with the choice of the equation of state in the form:

$$p = \frac{1 + \omega}{\omega} \rho. \quad (16)$$

Following the previous reasoning all we have to do is to solve equations (12)-(15), which taking into account (16), reduces to:

$$\alpha = C1 + \beta + \ln \phi, \quad (17)$$

$$\beta = C3 + C2 \ln r + (2 + 4\omega) \ln \phi, \quad (18)$$

$$\phi = r^{\frac{-3-C2-4\omega+k1}{4+7\omega}} \left(C4 + r^{\frac{2k1}{3+4\omega}} \right)^{-\frac{3+4\omega}{4+7\omega}} C5, \quad (19)$$

$$\rho = k2 \left(C4 + r^{\frac{2k1}{3+4\omega}} \right)^{\frac{(1+2\omega)(7+16\omega)}{(4+7\omega)}} r^{k3}, \quad (20)$$

where $C1, C2, C3, C4, C5$ is an arbitrary constants and :

$$\begin{aligned}
k1 &= \sqrt{C2^2 - 2C2(1 + 3\omega) + (3 + 4\omega)^2}, \\
k2 &= \frac{C4 \cdot e^{-2(C1+C3)} C5^{-5-8\omega} \omega \cdot k1^2}{\pi (3 + 4\omega) (4 + 7\omega)}, \\
k3 &= \frac{(1 + 2\omega) (3C2(3 + 4\omega) - (3 - k1 + 4\omega) (7 + 16\omega))}{(3 + 4\omega) (4 + 7\omega)}.
\end{aligned}$$

As a second example let us consider the equation of state which describes the space-time generated by a matter with $\rho = 0$ and $p \neq 0$. In this case, there is not analogous solutions in classical Einstein's gravity. The exact solution we can find using the value of $\omega = 0$. Then, from (2), (3) it follows that the sought-for line element, which describes the space-time generated in JBD theory, is given by

$$\alpha = C1 + \beta + \ln \phi, \quad (21)$$

$$\beta = C3 + C2 \ln r + 2 \ln \phi, \quad (22)$$

$$\phi = \frac{r^{\frac{1}{4}(-3-C2+k1)} C5}{\left(r^{\frac{2}{3}k1} + C4\right)^{\frac{3}{4}}}, \quad (23)$$

$$p = k2 \left(C4 + r^{\frac{2k1}{3}}\right)^{\frac{7}{4}} r^{k3}, \quad (24)$$

where $C1, C2, C3, C4, C5$ is an arbitrary constants and :

$$\begin{aligned}
k1 &= \sqrt{9 + C2(C2 - 2)}, \\
k2 &= \frac{C4 e^{-2(C1+C3)} k1^2}{12\pi C5^5}, \\
k3 &= \frac{21 - 9C2 - 7k1}{12}.
\end{aligned}$$

4 Conclusions

We have considered spherically symmetric interior solutions to the JBD field equations with a 2+1 number of spatial dimensions. The property of this field has been illustrated by computing the metric for the special equation of state (16). A reasonable method has been presented which allows one to solve, at least in this case, these equations. Moreover, using this method we find solution with $\rho = 0$ and $p \neq 0$ that has not analogous in classical Einstein's gravity. Finally, it is worth mentioning that the, in contrast with results of Bruckman and Kazes [7], relation between the scalar field ϕ and g_{00} is more complexity then $\phi = e^{k\beta}$.

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